Jana Nexus: Journal of Computer Science

ISSN(O): 3108-1916
Journal Homepage: jananexuscs.com
JN:JHST *Volume 01 Issue 12* *37-54*

| RESEARCH ARTICLE

Article DOI: 10.21474/JNCS01/114
DOI URL: http://dx.doi.org/10.21474/JNCS01/114

A SECOND DERIVATIVE HYBRID BLOCK METHOD WITH VARIATIONAL STABILITY PARAMETERS FOR THE DIRECT SOLUTION OF SECOND-ORDER INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

Mohammadesmail Nikfar

PhD in Pure Mathematics, Mathematics Teacher, Department of Education, District 4 Tehran, Iran.

Corresponding Author: Mohammadesmail Nikfar

| ABSTRACT

This paper presents the derivation and implementation of a second-derivative hybrid block method with stability parameters for the direct solution of second order initial value problems in ordinary differential equations. The method is developed using interpolation and collocation techniques to obtain continuous formulation, which is evaluated at selected grid and off-grid points to generate the discrete block scheme.

| KEYWORDS
Block method, Interpolation, Collocation, Hybrid and Polynomial basis function.

|ARTICLE INFORMATION

RECEIVED: 25 October 2025 **ACCEPTED:** 30 November 2025 **PUBLISHED:** December 2025

Abstract:-
This paper presents the derivation and implementation of a second-derivative hybrid block method with stability parameters for the direct solution of second order initial value problems in ordinary differential equations. The method is developed using interpolation and collocation techniques to obtain continuous formulation, which is evaluated at selected grid and off-grid points to generate the discrete block scheme. The convergence and stability properties of the method are investigated, and the results show that the method possesses A(α)-stability, A-stability, and additional regions of absolute stability for different values of the stability parameters. The method is applied to some selected initial value problems, and the results overlap with the exact solution, as illustrated in Figures.4, and.5 respectively. These results demonstrate the accuracy and efficiency of the proposed method when applied to both linear and nonlinear problems.

Introduction:-
This paper considers Initial Value Problems (IVPs) in Ordinary Differential Equations (ODEs) of the form:

$$y'' = f(x, y, y');\ y'(x_0) = y'_0, \qquad y(x_0) = y_0 x \in [x_0, x_n] \quad (1.1)$$

Equation (1.1) is called second order initial value problem in ordinary differential equation. Finding the direct numerical solution to equation of this form is an interesting area for researchers in literature.Traditionally equation (1.1) together with its associated conditions can be solved by reducing it to its first order equivalent (Dahlquist 1956). The reduced form is then solved using any appropriate numerical method. (Awoyemi 1999), this approach does not fully utilize all the information connected with certain ordinary differential equations, such as oscillatory nature of solution. (Jain 2012) and (Sair 2014)reported that this approach enlarges the size of equations and also causes computational burden in writing computer programmes which may jeopardize the performance of the method in terms of time and error. Hence (Areo and Adeniyi 2014), (Okuonghae and Ikhile 2014), (Bolaji 2017),(Adeyeyeand Omar 2017), (Adeyefa & Olagunju 2021), (Kamarun et al 2023), (Taiwo, et al 2023), (Abdulazeez 2024),(Workineh 2024), (Tadema 2024)etc. worked on different methods that solved equation (1.1)but considered fixed stability parameters. However, this article considers the derivation of hybrid block method with variational stability parameters which will help in improving the stability of the method.

Derivation of The Method:-
Consider the polynomial function as:-

$$p(x) = \sum_{j=0}^{n} a_j x^j \tag{2.1}$$

and use it to deduce an approximate solution to (1.1) as:

$$y(x) \cong p(x) = \sum_{j=0}^{q+t-1} a_j x^j \tag{2.2}$$

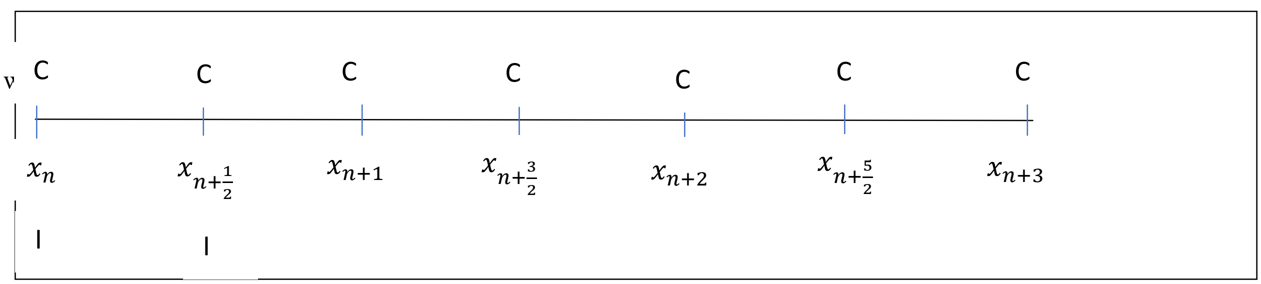
where a_j, $j = 0, 1, \dots$ are unknown parameters to be determined, $n = q + t - 1$, q *and* t are the points of interpolation and collocation, respectively. Then, the second derivative of (2.2):

$$y''(x) = f(x, y, y') = \sum_{j=1}^{q+t-1} j(j-1) a_j x_{n+i}^{j-2}\ i = \left(0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right) \tag{2.3}$$

$$y_{n+i} = \sum_{j=0}^{q+t-1} a_j x_{n+i}^{j}\ i = (0, \frac{1}{2}) \tag{2.4}$$

$$f_{n+i} = \sum_{j=1}^{q+t-1} j a_j x_{n+i}^{j-1}\ i = (0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3) \tag{2.5}$$

The interpolation of equation (2.4) at the point $x = x_n, x_{n+\frac{1}{2}}$ and collocation of equation (2.5) at $x = (0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3)$ for $q = 2$ *and* $t = 7$ i.e.



$$\begin{aligned}
a_0 &+ \frac{ha_1}{2} + \frac{h^2a_2}{4} + \frac{h^3a_3}{8} + \frac{h^4a_4}{16} + \frac{h^5a_5}{32} + \frac{h^6a_6}{64} + \frac{h^7a_7}{128} + \frac{h^8a_8}{256} + a_1x_n + ha_2x_n + \frac{3}{4}h^2a_3x_n + \frac{1}{2}h^3a_4x_n + \frac{5}{16}h^4a_5x_n \\
&+ \frac{3}{16}h^5a_6x_n + \frac{7}{64}h^6a_7x_n + \frac{1}{16}h^7a_8x_n + a_2x_n^2 + \frac{3}{2}ha_3x_n^2 + \frac{3}{2}h^2a_4x_n^2 + \frac{5}{4}h^3a_5x_n^2 + \frac{15}{16}h^4a_6x_n^2 \\
&+ \frac{21}{32}h^5a_7x_n^2 + \frac{7}{16}h^6a_8x_n^2 + a_3x_n^3 + 2ha_4x_n^3 + \frac{5}{2}h^2a_5x_n^3 + \frac{5}{2}h^3a_6x_n^3 + \frac{35}{16}h^4a_7x_n^3 + \frac{7}{4}h^5a_8x_n^3 + a_4x_n^4 \\
&+ \frac{5}{2}ha_5x_n^4 + \frac{15}{4}h^2a_6x_n^4 + \frac{35}{8}h^3a_7x_n^4 + \frac{35}{8}h^4a_8x_n^4 + a_5x_n^5 + 3ha_6x_n^5 + \frac{21}{4}h^2a_7x_n^5 + 7h^3a_8x_n^5 + a_6x_n^6 \\
&+ \frac{7}{2}ha_7x_n^6 + 7h^2a_8x_n^6 + a_7x_n^7 + 4ha_8x_n^7 + a_8x_n^8 \quad = \quad y_{n+\frac{1}{2}}
\end{aligned}$$

$$2a_2 + 6a_3x_n + 12a_4x_n^2 + 20a_5x_n^3 + 30a_6x_n^4 + 42a_7x_n^5 + 56a_8x_n^6 \qquad = \quad f_n$$

$$2a_2 + 3ha_3 + 3h^2a_4 + \frac{5h^3a_5}{2} + \frac{15h^4a_6}{8} + \frac{21h^5a_7}{16} + \frac{7h^6a_8}{8} + 6a_3x_n + 12ha_4x_n + 15h^2a_5x_n + 15h^3a_6x_n + \frac{105}{8}h^4a_7x_n + \frac{21}{2}h^5a_8x_n + 12a_4x_n^2 + 30ha_5x_n^2 + 45h^2a_6x_n^2 + \frac{105}{2}h^3a_7x_n^2 + \frac{105}{2}h^4a_8x_n^2 + 20a_5x_n^3 + 60ha_6x_n^3 + 105h^2a_7x_n^3 + 140h^3a_8x_n^3 + 30a_6x_n^4 + 105ha_7x_n^4 + 210h^2a_8x_n^4 + 42a_7x_n^5 + 168ha_8x_n^5 + 56a_8x_n^6 = f_{n+\frac{1}{2}}$$

$$2a_2 + 6ha_3 + 12h^2a_4 + 20h^3a_5 + 30h^4a_6 + 42h^5a_7 + 56h^6a_8 + 6a_3x_n + 24ha_4x_n + 60h^2a_5x_n + 120h^3a_6x_n + 210h^4a_7x_n + 336h^5a_8x_n + 12a_4x_n^2 + 60ha_5x_n^2 + 180h^2a_6x_n^2 + 420h^3a_7x_n^2 + 840h^4a_8x_n^2 + 20a_5x_n^3 + 120ha_6x_n^3 + 420h^2a_7x_n^3 + 1120h^3a_8x_n^3 + 30a_6x_n^4 + 210ha_7x_n^4 + 840h^2a_8x_n^4 + 42a_7x_n^5 + 336ha_8x_n^5 + 56a_8x_n^6 = f_{n+1}$$

$$2a_2 + 9ha_3 + 27h^2a_4 + \frac{135h^3a_5}{2} + \frac{1215h^4a_6}{8} + \frac{5103h^5a_7}{16} + \frac{5103h^6a_8}{8} + 6a_3x_n + 36ha_4x_n + 135h^2a_5x_n + 405h^3a_6x_n + \frac{8505}{8}h^4a_7x_n + \frac{5103}{2}h^5a_8x_n + 12a_4x_n^2 + 90ha_5x_n^2 + 405h^2a_6x_n^2 + \frac{2835}{2}h^3a_7x_n^2 + \frac{8505}{2}h^4a_8x_n^2 + 20a_5x_n^3 + 180ha_6x_n^3 + 945h^2a_7x_n^3 + 3780h^3a_8x_n^3 + 30a_6x_n^4 + 315ha_7x_n^4 + 1890h^2a_8x_n^4 + 42a_7x_n^5 + 504ha_8x_n^5 + 56a_8x_n^6 = f_{n+\frac{3}{2}}$$

$$2a_2 + 12ha_3 + 48h^2a_4 + 160h^3a_5 + 480h^4a_6 + 1344h^5a_7 + 3584h^6a_8 + 6a_3x_n + 48ha_4x_n + 240h^2a_5x_n + 960h^3a_6x_n + 3360h^4a_7x_n + 10752h^5a_8x_n + 12a_4x_n^2 + 120ha_5x_n^2 + 720h^2a_6x_n^2 + 3360h^3a_7x_n^2 + 13440h^4a_8x_n^2 + 20a_5x_n^3 + 240ha_6x_n^3 + 1680h^2a_7x_n^3 + 8960h^3a_8x_n^3 + 30a_6x_n^4 + 420ha_7x_n^4 + 3360h^2a_8x_n^4 + 42a_7x_n^5 + 672ha_8x_n^5 + 56a_8x_n^6 = f_{n+2}$$

$$2a_2 + 15ha_3 + 75h^2a_4 + \frac{625h^3a_5}{2} + \frac{9375h^4a_6}{8} + \frac{65625h^5a_7}{16} + \frac{109375h^6a_8}{8} + 6a_3x_n + 60ha_4x_n + 375h^2a_5x_n + 1875h^3a_6x_n + \frac{65625}{8}h^4a_7x_n + \frac{65625}{2}h^5a_8x_n + 12a_4x_n^2 + 150ha_5x_n^2 + 1125h^2a_6x_n^2 + \frac{13125}{2}h^3a_7x_n^2 + \frac{65625}{2}h^4a_8x_n^2 + 20a_5x_n^3 + 300ha_6x_n^3 + 2625h^2a_7x_n^3 + 17500h^3a_8x_n^3 + 30a_6x_n^4 + 525ha_7x_n^4 + 5250h^2a_8x_n^4 + 42a_7x_n^5 + 840ha_8x_n^5 + 56a_8x_n^6 = f_{n+\frac{5}{2}}$$

$$2a_2 + 18ha_3 + 108h^2a_4 + 540h^3a_5 + 2430h^4a_6 + 10206h^5a_7 + 40824h^6a_8 + 6a_3x_n + 72ha_4x_n + 540h^2a_5x_n + 3240h^3a_6x_n + 17010h^4a_7x_n + 81648h^5a_8x_n + 12a_4x_n^2 + 180ha_5x_n^2 + 1620h^2a_6x_n^2 + 11340h^3a_7x_n^2 + 68040h^4a_8x_n^2 + 20a_5x_n^3 + 360ha_6x_n^3 + 3780h^2a_7x_n^3 + 30240h^3a_8x_n^3 + 30a_6x_n^4 + 630ha_7x_n^4 + 7560h^2a_8x_n^4 + 42a_7x_n^5 + 1008ha_8x_n^5 + 56a_8x_n^6 = f_{n+3}$$

Noms of the above expansion are collected and presented in matrix form as:

$$
\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\
1 & \left(x_n+\frac{h}{2}\right) & \left(x_n+\frac{h}{2}\right)^2 & \left(x_n+\frac{h}{2}\right)^3 & \left(x_n+\frac{h}{2}\right)^4 & \left(x_n+\frac{h}{2}\right)^5 & \left(x_n+\frac{h}{2}\right)^6 & \left(x_n+\frac{h}{2}\right)^7 & \left(x_n+\frac{h}{2}\right)^8 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\
0 & 0 & 2 & 6(x_n+\frac{h}{2}) & 12(x_n+\frac{h}{2})^2 & 20(x_n+\frac{h}{2})^3 & 30(x_n+\frac{h}{2})^4 & 42(x_n+\frac{h}{2})^5 & 56(x_n+\frac{h}{2})^6 \\
0 & 1 & 2 & 6(x_n+h) & 12(x_n+h)^2 & 20(x_n+h)^3 & 30(x_n+h)^4 & 42(x_n+h)^5 & 56(x_n+h)^6 \\
0 & 1 & 2 & 6(x_n+\frac{3h}{2}) & 12(x_n+\frac{3h}{2})^2 & 20(x_n+\frac{3h}{2})^3 & 30(x_n+\frac{3h}{2})^4 & 42(x_n+\frac{3h}{2})^5 & 56(x_n+\frac{3h}{2})^6 \\
0 & 1 & 2 & 6(x_n+2h) & 12(x_n+2h)^2 & 20(x_n+2h)^3 & 30(x_n+2h)^4 & 42(x_n+2h)^5 & 56(x_n+2h)^6 \\
0 & 1 & 2 & 6(x_n+\frac{5h}{2}) & 12(x_n+\frac{5h}{2})^2 & 20(x_n+\frac{5h}{2})^3 & 30(x_n+\frac{5h}{2})^4 & 42(x_n+\frac{5h}{2})^5 & 56(x_n+\frac{5h}{2})^6 \\
0 & 1 & 2 & 6(x_n+3h) & 12(x_n+3h)^2 & 20(x_n+3h)^3 & 30(x_n+3h)^4 & 42(x_n+3h)^5 & 56(x_n+3h)^6
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}
= \begin{bmatrix} y_n \\ y_{n+\frac{1}{2}} \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{bmatrix} \qquad (2.6)
$$

Solving (2.6) using Gaussian elimination method yields:

$$
y(x) = \alpha_0 y_n + \alpha_{\frac{1}{2}} y_{n+\frac{1}{2}} + h^2\left[\beta_0 f_n + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_1 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta_3 f_{n+3}\right] \qquad (2.7)
$$

Where:

$$
\left.
\begin{aligned}
\alpha_0 &= 1-2t \\
\alpha_{\frac{1}{2}} &= 2t \\
\beta_0 &= \frac{h^2 t(-1+2t)\left(28549+2t\left(-31931+2t\left(17461+2t\left(-5275+2t(899+6t(-27+2t))\right)\right)\right)\right)}{241920} \\
\beta_{\frac{1}{2}} &= -\frac{h^2 t\left(9625+64(-4+t)t^2\left(315+2\ \left(-189+t\left(105+t(-28+3\ )\right)\right)\right)\right)}{40320} \\
\beta_1 &= \frac{h^2 t\left(17151+32t^2\left(-6300+t\left(12285+t\left(-9681+4t(959+5\ (-38+3\ ))\right)\right)\right)\right)}{80640} \\
\beta_{\frac{3}{2}} &= -\frac{h^2 t\left(10621+64t^2\left(-2100+t\left(4445+2t\left(-1953+t(847+15(-12+t)t)\right)\right)\right)\right)}{60480} \\
\beta_2 &= \frac{h^2 t\left(7703+32t^2\left(-3150+t\left(6930+t\left(-6447+4t(749+5t(-34+3t))\right)\right)\right)\right)}{80640} \\
\beta_{\frac{5}{2}} &= -\frac{h^2 t\left(1209+64t^2\left(-252+t\left(567+2t\left(-273+t\left(133+t(-32+3t)\right)\right)\right)\right)\right)}{40320} \\
\beta_3 &= \frac{h^2 t(995+32(-3+t)t^2(140+t(-273+4t(56+3(-7+t)t))))}{241920}
\end{aligned}
\right\} \quad (2.8)
$$

$$
Evaluating\ \alpha_0(t),\quad \alpha_{\frac{1}{2}}(t),\quad \beta_0(t),\quad \beta_{\frac{1}{2}}(t),\quad \beta_1(t),\quad \beta_{\frac{3}{2}}(t), \beta_2(t),\quad \beta_{\frac{5}{2}}(t),\quad \beta_3(t) at\ the\ non-interpolating\ points\ i.e. at\ x_{n+1}\ ,\ x_{n+\frac{3}{2}},\ x_{n+2},\ x_{n+\frac{5}{2}},\ x_{n+3}
$$

Using the transformation:

$x = ht + x_n$ (2.9)

Now replace x in equation (2.9) with each of the collocating points to obtain the corresponding values of t

For x_{n+1}

$$x_n + h = ht + x_n \Longrightarrow h = th \Longrightarrow t = 1$$

For $x_{n+\frac{3}{2}}$

$$x_n + \frac{3h}{2} = ht + x_n \Longrightarrow \frac{3h}{2} = ht \Longrightarrow t = \frac{3}{2}$$

For x_{n+2}

$x_n + 2h = ht + x_n \Longrightarrow 2h = ht \Longrightarrow t = 2$

For $x_{n+\frac{5}{2}}$

$$x_n + \frac{5h}{2} = ht + x_n \Longrightarrow \frac{5h}{2} = ht \Longrightarrow t = \frac{5}{2}$$

For x_{n+3}

$x_n + 3h = ht + x_n \Longrightarrow 3h = ht \Longrightarrow t = 3$

$i.e.\ t = 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ produces the following schemes:

$$\left.\begin{aligned}
y_{n+1} &= -y_n + 2y_{\frac{1}{2}+n} + \frac{863h^2 f_n}{48384} + \frac{8999h^2 f_{\frac{1}{2}+n}}{40320} - \frac{769h^2 f_{1+n}}{80640} + \frac{1987h^2 f_{\frac{3}{2}+n}}{60480} - \frac{1609h^2 f_{2+n}}{80640} + \frac{263h^2 f_{\frac{5}{2}+n}}{40320} - \frac{221h^2 f_{3+n}}{241920} \\
y_{n+\frac{3}{2}} &= -2y_n + 3y_{\frac{1}{2}+n} + \frac{2803h^2 f_n}{80640} + \frac{1265h^2 f_{\frac{1}{2}+n}}{2688} + \frac{1657h^2 f_{1+n}}{8960} + \frac{1777h^2 f_{\frac{3}{2}+n}}{20160} - \frac{1049h^2 f_{2+n}}{26880} + \frac{11}{896}h^2 f_{\frac{5}{2}+n} - \frac{137h^2 f_{3+n}}{80640} \\
y_{n+2} &= -3y_n + 4y_{\frac{1}{2}+n} + \frac{2089h^2 f_n}{40320} + \frac{4813h^2 f_{\frac{1}{2}+n}}{6720} + \frac{5461h^2 f_{1+n}}{13440} + \frac{3457h^2 f_{\frac{3}{2}+n}}{10080} - \frac{419h^2 f_{2+n}}{13440} + \frac{109h^2 f_{\frac{5}{2}+n}}{6720} - \frac{19h^2 f_{3+n}}{8064} \\
y_{n+\frac{5}{2}} &= -4y_n + 5y_{\frac{1}{2}+n} + \frac{1669h^2 f_n}{24192} + \frac{3875h^2 f_{\frac{1}{2}+n}}{4032} + \frac{5069h^2 f_{1+n}}{8064} + \frac{3751h^2 f_{\frac{3}{2}+n}}{6048} + \frac{1457h^2 f_{2+n}}{8064} + \frac{179h^2 f_{\frac{5}{2}+n}}{4032} - \frac{95h^2 f_{3+n}}{24192} \\
y_{n+3} &= -5y_n + 6y_{\frac{1}{2}+n} + \frac{1375h^2 f_n}{16128} + \frac{3259h^2 f_{\frac{1}{2}+n}}{2688} + \frac{1489h^2 f_{1+n}}{1792} + \frac{3751h^2 f_{\frac{3}{2}+n}}{4032} + \frac{2059h^2 f_{2+n}}{5376} + \frac{265}{896}h^2 f_{\frac{5}{2}+n} + \frac{199h^2 f_{3+n}}{16128}
\end{aligned}\right\} \quad (2.10)$$

Differentiating (2.8)

$$\left.\begin{aligned}
\alpha_0' &= -2 \\
\alpha_{\frac{1}{2}}' &= 2 \\
\beta_0' &= -\frac{28549h^2}{241920} + h^2 t - \frac{49h^2t^2}{20} + \frac{406h^2t^3}{135} - \frac{49h^2t^4}{24} + \frac{7h^2t^5}{9} - \frac{7h^2t^6}{45} + \frac{4h^2t^7}{315} \\
\beta_{\frac{1}{2}}' &= -\frac{275h^2}{1152} + 6h^2t^2 - \frac{58h^2t^3}{5} + \frac{29h^2t^4}{3} - \frac{62h^2t^5}{15} + \frac{8h^2t^6}{9} - \frac{8h^2t^7}{105} \\
\beta_1' &= \frac{5717h^2}{26880} - \frac{15h^2t^2}{2} + \frac{39h^2t^3}{2} - \frac{461h^2t^4}{24} + \frac{137h^2t^5}{15} - \frac{19h^2t^6}{9} + \frac{4h^2t^7}{21} \\
\beta_{\frac{3}{2}}' &= -\frac{10621h^2}{60480} + \frac{20h^2t^2}{3} - \frac{508h^2t^3}{27} + \frac{62h^2t^4}{3} - \frac{484h^2t^5}{45} + \frac{8h^2t^6}{3} - \frac{16h^2t^7}{63} \\
\beta_2' &= \frac{7703h^2}{80640} - \frac{15h^2t^2}{4} + 11h^2t^3 - \frac{307h^2t^4}{24} + \frac{107h^2t^5}{15} - \frac{17h^2t^6}{9} + \frac{4h^2t^7}{21} \\
\beta_{\frac{5}{2}}' &= -\frac{403h^2}{13440} + \frac{6h^2t^2}{5} - \frac{18h^2t^3}{5} + \frac{13h^2t^4}{3} - \frac{38h^2t^5}{15} + \frac{32h^2t^6}{45} - \frac{8h^2t^7}{105} \\
\beta_3' &= \frac{199h^2}{48384} - \frac{h^2t^2}{6} + \frac{137h^2t^3}{270} - \frac{5h^2t^4}{8} + \frac{17h^2t^5}{45} - \frac{h^2t^6}{9} + \frac{4h^2t^7}{315}
\end{aligned}\right\} \quad (2.11)$$

Equation (2.11) is evaluated at all the grid points i.e. at $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ using the derivative of (2.7) i.e.

$$hy_n' = \alpha_0' y_n + \alpha_{\frac{1}{2}}' y_{n+\frac{1}{2}} + h^2 \left[\beta_0' f_n + \beta_{\frac{1}{2}}' f_{n+\frac{1}{2}} + \beta_1' f_{n+1} + \beta_{\frac{3}{2}}' f_{n+\frac{3}{2}} + \beta_2' f_{n+2} + \beta_{\frac{5}{2}}' f_{n+\frac{5}{2}} + \beta_3' f_{n+3}\right] \quad (2.12)$$

which gives:

$$\left.\begin{aligned}
hy'_n &= -2y_n + 2y_{n+\frac{1}{2}} + h^2\left[-\frac{28549}{241920}f_n - \frac{275}{1152}f_{n+\frac{1}{2}} + \frac{5717}{26880}f_{n+1} - \frac{10621}{60480}f_{n+\frac{3}{2}} + \frac{7703}{80640}f_{n+2} - \frac{403}{13440}f_{n+\frac{5}{2}} + \frac{199}{48384}f_{n+3}\right]\\
hy'_{n+\frac{1}{2}} &= -2y_n + 2y_{n+\frac{1}{2}} + h^2\left[\frac{275}{6912}f_n + \frac{12079}{40320}f_{n+\frac{1}{2}} - \frac{13823}{80640}f_{n+1} + \frac{8131}{60480}f_{n+\frac{3}{2}} - \frac{5771}{80640}f_{n+2} + \frac{179}{8064}f_{n+\frac{5}{2}} - \frac{731}{241920}f_{n+3}\right]\\
hy'_{n+1} &= -2y_n + 2y_{n+\frac{1}{2}} + h^2\left[\frac{2633}{80640}f_n + \frac{4091}{8064}f_{n+\frac{1}{2}} + \frac{17503}{80640}f_{n+1} + \frac{1}{20160}f_{n+\frac{3}{2}} - \frac{181}{16128}f_{n+2} + \frac{199}{40320}f_{n+\frac{5}{2}} - \frac{1}{1280}f_{n+3}\right]\\
hy'_{n+\frac{3}{2}} &= -2y_n + 2y_{n+\frac{1}{2}} + h^2\left[\frac{8441}{241920}f_n + \frac{3907}{8064}f_{n+\frac{1}{2}} + \frac{12683}{26880}f_{n+1} + \frac{3751}{12096}f_{n+\frac{3}{2}} - \frac{5419}{80640}f_{n+2} + \frac{7}{384}f_{n+\frac{5}{2}} - \frac{571}{241920}f_{n+3}\right]\\
hy'_{n+2} &= -2y_n + 2y_{n+\frac{1}{2}} + h^2\left[\frac{8059}{241920}f_n + \frac{20071}{40320}f_{n+\frac{1}{2}} + \frac{6707}{16128}f_{n+1} + \frac{37507}{60480}f_{n+\frac{3}{2}} + \frac{2161}{11520}f_{n+2} - \frac{37}{8064}f_{n+\frac{5}{2}} - \frac{29}{241920}f_{n+3}\right]\\
hy'_{n+\frac{5}{2}} &= -2y_n + 2y_{n+\frac{1}{2}} + h^2\left[\frac{2867}{80640}f_n + \frac{3875}{8064}f_{n+\frac{1}{2}} + \frac{38401}{80640}f_{n+1} + \frac{1399}{2880}f_{n+\frac{3}{2}} + \frac{46453}{80640}f_{n+2} + \frac{8191}{40320}f_{n+\frac{5}{2}} - \frac{13}{1792}f_{n+3}\right]\\
hy'_{n+3} &= -2y_n + 2y_{n+\frac{1}{2}} + h^2\left[\frac{1375}{48384}f_n + \frac{21479}{40320}f_{n+\frac{1}{2}} + \frac{1187}{3840}f_{n+1} + \frac{48131}{60480}f_{n+\frac{3}{2}} + \frac{15479}{80640}f_{n+2} + \frac{1993}{2688}f_{n+\frac{5}{2}} + \frac{36419}{241920}f_{n+3}\right]
\end{aligned}\right\}\quad(2.13)$$

Now combining Equation (2.10) and (2.13) gives:

$$\left.\begin{aligned}
y_{n+1} - 2y_{\frac{1}{2}+n} &= -y_n + \frac{863h^2f_n}{48384} + \frac{8999h^2f_{\frac{1}{2}+n}}{40320} - \frac{769h^2f_{1+n}}{80640} + \frac{1987h^2f_{\frac{3}{2}+n}}{60480} - \frac{1609h^2f_{2+n}}{80640} + \frac{263h^2f_{\frac{5}{2}+n}}{40320} - \frac{221h^2f_{3+n}}{241920}\\
y_{n+\frac{3}{2}} - 3y_{\frac{1}{2}+n} &= -2y_n + \frac{2803h^2f_n}{80640} + \frac{1265h^2f_{\frac{1}{2}+n}}{2688} + \frac{1657h^2f_{1+n}}{8960} + \frac{1777h^2f_{\frac{3}{2}+n}}{20160} - \frac{1049h^2f_{2+n}}{26880} + \frac{11}{896}h^2f_{\frac{5}{2}+n} - \frac{137h^2f_{3+n}}{80640}\\
y_{n+2} - 4y_{\frac{1}{2}+n} &= -3y_n + \frac{2089h^2f_n}{40320} + \frac{4813h^2f_{\frac{1}{2}+n}}{6720} + \frac{5461h^2f_{1+n}}{13440} + \frac{3457h^2f_{\frac{3}{2}+n}}{10080} - \frac{419h^2f_{2+n}}{13440} + \frac{109h^2f_{\frac{5}{2}+n}}{6720} - \frac{19h^2f_{3+n}}{8064}\\
y_{n+\frac{5}{2}} - 5y_{\frac{1}{2}+n} &= -4y_n + \frac{1669h^2f_n}{24192} + \frac{3875h^2f_{\frac{1}{2}+n}}{4032} + \frac{5069h^2f_{1+n}}{8064} + \frac{3751h^2f_{\frac{3}{2}+n}}{6048} + \frac{1457h^2f_{2+n}}{8064} + \frac{179h^2f_{\frac{5}{2}+n}}{4032} - \frac{95h^2f_{3+n}}{24192}\\
y_{n+3} - 6y_{\frac{1}{2}+n} &= -5y_n + \frac{1375h^2f_n}{16128} + \frac{3259h^2f_{\frac{1}{2}+n}}{2688} + \frac{1489h^2f_{1+n}}{1792} + \frac{3751h^2f_{\frac{3}{2}+n}}{4032} + \frac{2059h^2f_{2+n}}{5376} + \frac{265}{896}h^2f_{\frac{5}{2}+n} + \frac{199h^2f_{3+n}}{16128}\\
hy'_n - 2y_{n+\frac{1}{2}} + 2y_n &= -\frac{28549h^2}{241920}f_n - \frac{275h^2}{1152}f_{n+\frac{1}{2}} + \frac{5717h^2}{26880}f_{n+1} - \frac{10621h^2}{60480}f_{n+\frac{3}{2}} + \frac{7703h^2}{80640}f_{n+2} - \frac{403h^2}{13440}f_{n+\frac{5}{2}} + \frac{199h^2}{48384}f_{n+3}\\
hy'_{n+\frac{1}{2}} - 2y_{n+\frac{1}{2}} + 2y_n &= \frac{275h^2}{6912}f_n + \frac{12079h^2}{40320}f_{n+\frac{1}{2}} - \frac{13823h^2}{80640}f_{n+1} + \frac{8131h^2}{60480}f_{n+\frac{3}{2}} - \frac{5771h^2}{80640}f_{n+2} + \frac{179h^2}{8064}f_{n+\frac{5}{2}} - \frac{731h^2}{241920}f_{n+3}\\
hy'_{n+1} - 2y_{n+\frac{1}{2}} + 2y_n &= \frac{2633h^2}{80640}f_n + \frac{4091h^2}{8064}f_{n+\frac{1}{2}} + \frac{17503h^2}{80640}f_{n+1} + \frac{h^2}{20160}f_{n+\frac{3}{2}} - \frac{181h^2}{16128}f_{n+2} + \frac{199h^2}{40320}f_{n+\frac{5}{2}} - \frac{h^2}{1280}f_{n+3}\\
hy'_{n+\frac{3}{2}} - 2y_{n+\frac{1}{2}} + 2y_n &= \frac{8441h^2}{241920}f_n + \frac{3907h^2}{8064}f_{n+\frac{1}{2}} + \frac{12683h^2}{26880}f_{n+1} + \frac{3751h^2}{12096}f_{n+\frac{3}{2}} - \frac{5419h^2}{80640}f_{n+2} + \frac{7h^2}{384}f_{n+\frac{5}{2}} - \frac{571h^2}{241920}f_{n+3}\\
hy'_{n+2} - 2y_{n+\frac{1}{2}} + 2y_n &= \frac{8059h^2}{241920}f_n + \frac{20071h^2}{40320}f_{n+\frac{1}{2}} + \frac{6707h^2}{16128}f_{n+1} + \frac{37507h^2}{60480}f_{n+\frac{3}{2}} + \frac{2161h^2}{11520}f_{n+2} - \frac{37h^2}{8064}f_{n+\frac{5}{2}} - \frac{29h^2}{241920}f_{n+3}\\
hy'_{n+\frac{5}{2}} - 2y_{n+\frac{1}{2}} + 2y_n &= \frac{2867h^2}{80640}f_n + \frac{3875h^2}{8064}f_{n+\frac{1}{2}} + \frac{38401h^2}{80640}f_{n+1} + \frac{1399h^2}{2880}f_{n+\frac{3}{2}} + \frac{46453h^2}{80640}f_{n+2} + \frac{8191h^2}{40320}f_{n+\frac{5}{2}} - \frac{13h^2}{1792}f_{n+3}\\
hy'_{n+3} - 2y_{n+\frac{1}{2}} + 2y_n &= \frac{1375h^2}{48384}f_n + \frac{21479h^2}{40320}f_{n+\frac{1}{2}} + \frac{1187h^2}{3840}f_{n+1} + \frac{48131h^2}{60480}f_{n+\frac{3}{2}} + \frac{15479h^2}{80640}f_{n+2} + \frac{1993h^2}{2688}f_{n+\frac{5}{2}} + \frac{36419h^2}{241920}f_{n+3}
\end{aligned}\right\}\quad(2.14)$$

Equation (2.14) can be written in block form as:

$$
\begin{bmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0\\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 & 0\\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0\\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0\\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0\\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{2}}\\ y_{n+1}\\ y_{n+\frac{3}{2}}\\ y_{n+2}\\ y_{n+\frac{5}{2}}\\ y_{n+3}\\ y'_{n+\frac{1}{2}}\\ y'_{n+1}\\ y'_{n+\frac{3}{2}}\\ y'_{n+2}\\ y'_{n+\frac{5}{2}}\\ y'_{n+3}
\end{bmatrix}
$$

$$
=\begin{bmatrix}
-1 & 0\\ -2 & 0\\ -3 & 0\\ -4 & 0\\ -5 & 0\\ -2 & -h\\ -2 & 0\\ -2 & 0\\ -2 & 0\\ -2 & 0\\ -2 & 0\\ -2 & 0
\end{bmatrix}
\begin{bmatrix} y_n\\ y'_n \end{bmatrix}
+
\begin{bmatrix}
\frac{863h^2}{48384}\\
\frac{2803h^2}{80640}\\
\frac{2089h^2}{40320}\\
\frac{1669h^2}{24192}\\
\frac{1375h^2}{16128}\\
-\frac{28549h^2}{241920}\\
\frac{275h^2}{6912}\\
\frac{2633h^2}{80640}\\
\frac{8441h^2}{241920}\\
\frac{8059h^2}{241920}\\
\frac{2867h^2}{80640}\\
\frac{1375h^2}{48384}
\end{bmatrix}
[f_n]
$$

$$
+h^2\begin{bmatrix}
\frac{8999}{40320} & -\frac{769}{80640} & \frac{1987}{60480} & -\frac{1609}{80640} & \frac{263}{40320} & -\frac{221}{241920}\\
\frac{1265}{2688} & \frac{1657}{8960} & \frac{1777}{20160} & -\frac{1049}{26880} & \frac{11}{896} & -\frac{137}{80640}\\
\frac{4813}{6720} & \frac{5461}{13440} & \frac{3457}{10080} & -\frac{419}{13440} & \frac{109}{6720} & -\frac{19}{8064}\\
\frac{3875}{4032} & \frac{5069}{8064} & \frac{3751}{6048} & \frac{1457}{8064} & \frac{179}{4032} & -\frac{95}{24192}\\
\frac{3259}{2688} & \frac{1489}{1792} & \frac{3751}{4032} & \frac{2059}{5376} & \frac{265}{896} & \frac{199}{16128}\\
-\frac{275}{1152} & \frac{5717}{26880} & -\frac{10621}{60480} & \frac{7703}{80640} & -\frac{403}{13440} & \frac{199}{48384}\\
\frac{12079}{40320} & -\frac{13823}{80640} & \frac{8131}{60480} & -\frac{5771}{80640} & \frac{179}{8064} & -\frac{731}{241920}\\
\frac{4091}{8064} & \frac{17503}{80640} & \frac{1}{20160} & -\frac{181}{16128} & \frac{199}{40320} & -\frac{1}{1280}\\
\frac{3907}{8064} & \frac{12683}{26880} & \frac{3751}{12096} & -\frac{5419}{80640} & \frac{7}{384} & -\frac{571}{241920}\\
\frac{20071}{40320} & \frac{6707}{16128} & \frac{37507}{60480} & \frac{2161}{11520} & -\frac{37}{8064} & -\frac{29}{241920}\\
\frac{3875}{8064} & \frac{38401}{80640} & \frac{1399}{2880} & \frac{46453}{80640} & \frac{8191}{40320} & -\frac{13}{1792}\\
\frac{21479}{40320} & \frac{1187}{3840} & \frac{48131}{60480} & \frac{15479}{80640} & \frac{1993}{2688} & \frac{36419}{241920}
\end{bmatrix}
\begin{bmatrix} f_{n+\frac{1}{2}}\\ f_{n+1}\\ f_{n+\frac{3}{2}}\\ f_{n+2}\\ f_{n+\frac{5}{2}}\\ f_{n+3}\end{bmatrix} \qquad (2.15)
$$

Now let:

$$
A^v=\begin{bmatrix}
-2&1&0&0&0&0&0&0&0&0&0&0\\
-3&0&1&0&0&0&0&0&0&0&0&0\\
-4&0&0&1&0&0&0&0&0&0&0&0\\
-5&0&0&0&1&0&0&0&0&0&0&0\\
-6&0&0&0&0&1&0&0&0&0&0&0\\
-2&0&0&0&0&0&0&0&0&0&0&0\\
-2&0&0&0&0&0&h&0&0&0&0&0\\
-2&0&0&0&0&0&0&h&0&0&0&0\\
-2&0&0&0&0&0&0&0&h&0&0&0\\
-2&0&0&0&0&0&0&0&0&h&0&0\\
-2&0&0&0&0&0&0&0&0&0&h&0\\
-2&0&0&0&0&0&0&0&0&0&0&h
\end{bmatrix};\ A^x=\begin{bmatrix}
-1&0\\ -2&0\\ -3&0\\ -4&0\\ -5&0\\ -2&-h\\ -2&0\\ -2&0\\ -2&0\\ -2&0\\ -2&0\\ -2&0
\end{bmatrix};\quad E^x=\begin{bmatrix}
\frac{863h^2}{48384}\\ \frac{2803h^2}{80640}\\ \frac{2089h^2}{40320}\\ \frac{1669h^2}{24192}\\ \frac{1375h^2}{16128}\\ -\frac{28549h^2}{241920}\\ \frac{275h^2}{6912}\\ \frac{2633h^2}{80640}\\ \frac{8441h^2}{241920}\\ \frac{8059h^2}{241920}\\ \frac{2867h^2}{80640}\\ \frac{1375h^2}{48384}
\end{bmatrix};
$$

$$E^{v}=\begin{bmatrix}
\frac{8999}{40320} & -\frac{769}{80640} & \frac{1987}{60480} & -\frac{1609}{80640} & \frac{263}{40320} & -\frac{221}{241920}\\
\frac{1265}{2688} & \frac{1657}{8960} & \frac{1777}{20160} & -\frac{1049}{26880} & \frac{11}{896} & -\frac{137}{80640}\\
\frac{4813}{6720} & \frac{5461}{13440} & \frac{3457}{10080} & -\frac{419}{13440} & \frac{109}{6720} & -\frac{19}{8064}\\
\frac{3875}{4032} & \frac{5069}{8064} & \frac{3751}{6048} & \frac{1457}{8064} & \frac{179}{4032} & -\frac{95}{24192}\\
\frac{3259}{2688} & \frac{1489}{1792} & \frac{3751}{4032} & \frac{2059}{5376} & \frac{265}{896} & \frac{199}{16128}\\
-\frac{275}{1152} & \frac{5717}{26880} & -\frac{10621}{60480} & \frac{7703}{80640} & -\frac{403}{13440} & \frac{199}{48384}\\
\frac{12079}{40320} & -\frac{13823}{80640} & \frac{8131}{60480} & -\frac{5771}{80640} & \frac{179}{8064} & -\frac{731}{241920}\\
\frac{4091}{8064} & \frac{17503}{80640} & \frac{1}{20160} & -\frac{181}{16128} & \frac{199}{40320} & -\frac{1}{1280}\\
\frac{3907}{8064} & \frac{12683}{26880} & \frac{3751}{12096} & -\frac{5419}{80640} & \frac{7}{384} & -\frac{571}{241920}\\
\frac{20071}{40320} & \frac{6707}{16128} & \frac{37507}{60480} & \frac{2161}{11520} & -\frac{37}{8064} & -\frac{29}{241920}\\
\frac{3875}{8064} & \frac{38401}{80640} & \frac{1399}{2880} & \frac{46453}{80640} & \frac{8191}{40320} & -\frac{13}{1792}\\
\frac{21479}{40320} & \frac{1187}{3840} & \frac{48131}{60480} & \frac{15479}{80640} & \frac{1993}{2688} & \frac{36419}{241920}
\end{bmatrix}$$

Multiply equation (2.15) by the inverse of A^{v} and it yielded:

$$
\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{2}}\\ y_{n+1}\\ y_{n+\frac{3}{2}}\\ y_{n+2}\\ y_{n+\frac{5}{2}}\\ y_{n+3}\\ y'_{n+\frac{1}{2}}\\ y'_{n+1}\\ y'_{n+\frac{3}{2}}\\ y'_{n+2}\\ y'_{n+\frac{5}{2}}\\ y'_{n+3}
\end{bmatrix}
=
$$

$$
\begin{bmatrix}
1 & \frac{h}{2}\\ 1 & h\\ 1 & \frac{3h}{2}\\ 1 & 2h\\ 1 & \frac{5h}{2}\\ 1 & 3h\\ 0&1\\ 0&1\\ 0&1\\ 0&1\\ 0&1\\ 0&1
\end{bmatrix}
\begin{bmatrix} y_n\\ y'_n \end{bmatrix}
+
\begin{bmatrix}
\frac{28549h^2}{483840}\\ \frac{1027h^2}{7560}\\ \frac{759h^2}{3584}\\ \frac{272h^2}{945}\\ \frac{35225h^2}{96768}\\ \frac{123h^2}{280}\\ \frac{19087h}{120960}\\ \frac{1139h}{7560}\\ \frac{137h}{896}\\ \frac{143h}{945}\\ \frac{3715h}{24192}\\ \frac{41h}{280}
\end{bmatrix}
[f_n]
+
\begin{bmatrix}
\frac{275h^2}{2304} & -\frac{5717h^2}{53760} & \frac{10621h^2}{120960} & -\frac{7703h^2}{161280} & \frac{403h^2}{26880} & -\frac{199h^2}{96768}\\
\frac{97h^2}{210} & -\frac{2h^2}{9} & \frac{197h^2}{945} & -\frac{97h^2}{840} & \frac{23h^2}{630} & -\frac{19h^2}{3780}\\
\frac{1485h^2}{1792} & -\frac{2403h^2}{17920} & \frac{45h^2}{128} & -\frac{3267h^2}{17920} & \frac{513h^2}{8960} & -\frac{141h^2}{17920}\\
\frac{376h^2}{315} & -\frac{2h^2}{105} & \frac{656h^2}{945} & -\frac{2h^2}{9} & \frac{8h^2}{105} & -\frac{2h^2}{189}\\
\frac{8375h^2}{5376} & \frac{3125h^2}{32256} & \frac{25625h^2}{24192} & -\frac{625h^2}{10752} & \frac{275h^2}{2304} & -\frac{1375h^2}{96768}\\
\frac{27h^2}{14} & \frac{27h^2}{140} & \frac{51h^2}{35} & \frac{27h^2}{280} & \frac{27h^2}{70} & 0\\
\frac{2713h}{5040} & -\frac{15487h}{40320} & \frac{293h}{945} & -\frac{6737h}{40320} & \frac{263h}{5040} & -\frac{863h}{120960}\\
\frac{47h}{63} & \frac{11h}{2520} & \frac{166h}{945} & -\frac{269h}{2520} & \frac{11h}{315} & -\frac{37h}{7560}\\
\frac{81h}{112} & \frac{1161h}{4480} & \frac{17h}{35} & -\frac{729h}{4480} & \frac{27h}{560} & -\frac{29h}{4480}\\
\frac{232h}{315} & \frac{64h}{315} & \frac{752h}{945} & \frac{29h}{315} & \frac{8h}{315} & -\frac{4h}{945}\\
\frac{725h}{1008} & \frac{2125h}{8064} & \frac{125h}{189} & \frac{3875h}{8064} & \frac{235h}{1008} & -\frac{275h}{24192}\\
\frac{27h}{35} & \frac{27h}{280} & \frac{34h}{35} & \frac{27h}{280} & \frac{27h}{35} & \frac{41h}{280}
\end{bmatrix}
\begin{bmatrix}
f_{n+\frac{1}{2}}\\ f_{n+1}\\ f_{n+\frac{3}{2}}\\ f_{n+2}\\ f_{n+\frac{5}{2}}\\ f_{n+3}
\end{bmatrix}
\tag{2.16}
$$

Expand (2.16) and it yielded:

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= y_n + \frac{h}{2}y'_n + \frac{28549h^2}{483840d}f_n + \frac{275h^2}{2304s}f_{n+\frac{1}{2}} - \frac{5717h^2}{53760s}f_{n+1} + \frac{10621h^2}{120960s}f_{n+\frac{3}{2}} - \frac{7703h^2}{161280s}f_{n+2} + \frac{403h^2}{26880s}f_{n+\frac{5}{2}} - \frac{199h^2}{96768s}f_{n+3} \\
y_{n+1} &= y_n + hy'_n + \frac{1027h^2}{7560d}f_n + \frac{97h^2}{210s}f_{n+\frac{1}{2}} - \frac{2h^2}{9s}f_{n+1} + \frac{197h^2}{945s}f_{n+\frac{3}{2}} - \frac{97h^2}{840s}f_{n+2} + \frac{23h^2}{630s}f_{n+\frac{5}{2}} - \frac{19h^2}{3780s}f_{n+3} \\
y_{n+\frac{3}{2}} &= y_n + \frac{3}{2}hy'_n + \frac{759h^2}{3584d}f_n + \frac{1485h^2}{1792s}f_{n+\frac{1}{2}} - \frac{2403h^2}{17920s}f_{n+1} + \frac{45h^2}{128s}f_{n+\frac{3}{2}} - \frac{3267h^2}{17920s}f_{n+2} + \frac{513h^2}{8960s}f_{n+\frac{5}{2}} - \frac{141h^2}{17920s}f_{n+3} \\
y_{n+2} &= y_n + 2hy'_n + \frac{272h^2}{945d}f_n + \frac{376h^2}{315s}f_{n+\frac{1}{2}} - \frac{2h^2}{105s}f_{n+1} + \frac{656h^2}{945s}f_{n+\frac{3}{2}} - \frac{2h^2}{9s}f_{n+2} + \frac{8h^2}{105s}f_{n+\frac{5}{2}} - \frac{2h^2}{189s}f_{n+3} \\
y_{n+\frac{5}{2}} &= y_n + \frac{5}{2}hy'_n + \frac{35225h^2}{96768d}f_n + \frac{8375h^2}{5376s}f_{n+\frac{1}{2}} + \frac{3125h^2}{32256s}f_{n+1} + \frac{25625h^2}{24192s}f_{n+\frac{3}{2}} - \frac{625h^2}{10752s}f_{n+2} + \frac{275h^2}{2304s}f_{n+\frac{5}{2}} - \frac{1375h^2}{96768s}f_{n+3} \\
y_{n+3} &= y_n + 3hy'_n + \frac{123h^2}{280d}f_n + \frac{27h^2}{14s}f_{n+\frac{1}{2}} + \frac{27h^2}{140s}f_{n+1} + \frac{51h^2}{35s}f_{n+\frac{3}{2}} + \frac{27h^2}{280s}f_{n+2} + \frac{27h^2}{70s}f_{n+\frac{5}{2}} \\
y'_{n+\frac{1}{2}} &= y'_n + \frac{19087h}{120960d}f_n + \frac{2713h}{5040s}f_{n+\frac{1}{2}} - \frac{15487h}{40320s}f_{n+1} + \frac{293h}{945s}f_{n+\frac{3}{2}} - \frac{6737h}{40320s}f_{n+2} + \frac{263h}{5040s}f_{n+\frac{5}{2}} - \frac{863h}{120960s}f_{n+3} \\
y'_{n+1} &= y'_n + \frac{1139h}{7560d}f_n + \frac{47h}{63s}f_{n+\frac{1}{2}} + \frac{11h}{2520s}f_{n+1} + \frac{166h}{945s}f_{n+\frac{3}{2}} - \frac{269h}{2520s} + \frac{11h}{315s}f_{n+\frac{5}{2}} - \frac{37h}{7560s}f_{n+3} \\
y'_{n+\frac{3}{2}} &= y'_n + \frac{137h}{896d}f_n + \frac{81h}{112s}f_{n+\frac{1}{2}} + \frac{1161h}{4480s}f_{n+1} + \frac{17h}{35s}f_{n+\frac{3}{2}} - \frac{729h}{4480s}f_{n+2} + \frac{27h}{560s}f_{n+\frac{5}{2}} - \frac{29h}{4480s}f_{n+3} \\
y'_{n+2} &= y'_n + \frac{143h}{945d}f_n + \frac{232h}{315s}f_{n+\frac{1}{2}} + \frac{64h}{315s}f_{n+1} + \frac{752h}{945s}f_{n+\frac{3}{2}} + \frac{29h}{315s}f_{n+2} + \frac{8h}{315s}f_{n+\frac{5}{2}} - \frac{4h}{945s}f_{n+3} \\
y'_{n+\frac{5}{2}} &= y'_n + \frac{3715h}{24192d}f_n + \frac{725h}{1008s}f_{n+\frac{1}{2}} + \frac{2125h}{8064s}f_{n+1} + \frac{125h}{189s}f_{n+\frac{3}{2}} + \frac{3875h}{8064s}f_{n+2} + \frac{235h}{1008s}f_{n+\frac{5}{2}} - \frac{275h}{24192s}f_{n+3} \\
y'_{n+3} &= y'_n + \frac{41h}{280d}f_n + \frac{27h}{35s}f_{n+\frac{1}{2}} + \frac{27h}{280s}f_{n+1} + \frac{34h}{35s}f_{n+\frac{3}{2}} + \frac{27h}{280s}f_{n+2} + \frac{27h}{35s}f_{n+\frac{5}{2}} + \frac{41h}{280s}f_{n+3}
\end{aligned}\right\} \quad (2.17)$$

Equation (2.17) can be written in block form as:

$$\left.\begin{aligned}
IY_{n+\frac{i}{2}} &= AY_{n-\frac{j}{2}} + hBY'_{n-\frac{j}{2}} + h^2\left(\frac{1}{s}CF_{n+\frac{i}{2}} + \frac{1}{d}NF_{n-\frac{j}{2}}\right) \\
GY'_{n+\frac{i}{2}} &= TY'_{n-\frac{j}{2}} + h\left(\frac{1}{s}EF_{n+\frac{i}{2}} + \frac{1}{d}VF_{n-\frac{j}{2}}\right)
\end{aligned}\right\} \quad (2.18)$$

Where ‘d’ and ‘s’ are the stability parameters.

$$I=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},A=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix},B=\begin{bmatrix}0&\frac{h}{2}&0&0&0&0\\0&h&0&0&0&0\\0&\frac{3h}{2}&0&0&0&0\\0&2h&0&0&0&0\\0&\frac{5h}{2}&0&0&0&0\\0&3h&0&0&0&0\end{bmatrix},C$$

$$=\frac{1}{s}\begin{bmatrix}\frac{275h^2}{2304}&-\frac{5717h^2}{53760}&\frac{10621h^2}{120960}&-\frac{7703h^2}{161280}&\frac{403h^2}{26880}&-\frac{199h^2}{96768}\\\frac{97h^2}{210}&-\frac{2h^2}{9}&\frac{197h^2}{945}&-\frac{97h^2}{840}&\frac{23h^2}{630}&-\frac{19h^2}{3780}\\\frac{1485h^2}{1792}&-\frac{2403h^2}{17920}&\frac{45h^2}{128}&-\frac{3267h^2}{17920}&\frac{513h^2}{8960}&-\frac{141h^2}{17920}\\\frac{376h^2}{315}&-\frac{2h^2}{105}&\frac{656h^2}{945}&-\frac{2h^2}{9}&\frac{8h^2}{105}&-\frac{2h^2}{189}\\\frac{8375h^2}{5376}&\frac{3125h^2}{32256}&\frac{25625h^2}{24192}&-\frac{625h^2}{10752}&\frac{275h^2}{2304}&-\frac{1375h^2}{96768}\\\frac{27h^2}{14}&\frac{27h^2}{140}&\frac{51h^2}{35}&\frac{27h^2}{280}&\frac{27h^2}{70}&0\end{bmatrix},N$$

$$=\frac{1}{d}\begin{bmatrix}0&0&0&0&0&\frac{28549h^2}{483840}\\0&0&0&0&0&\frac{1027h^2}{7560}\\0&0&0&0&0&\frac{759h^2}{3584}\\0&0&0&0&0&\frac{272h^2}{945}\\0&0&0&0&0&\frac{35225h^2}{96768}\\0&0&0&0&0&\frac{123h^2}{280}\end{bmatrix},$$

$$G=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},T=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix},E$$

$$=\frac{1}{s}\begin{bmatrix}\frac{2713h}{5040}&-\frac{15487h}{40320}&\frac{293h}{945}&-\frac{6737h}{40320}&\frac{263h}{5040}&-\frac{863h}{120960}\\\frac{47h}{63}&\frac{11h}{2520}&\frac{166h}{945}&-\frac{269h}{2520}&\frac{11h}{315}&-\frac{37h}{7560}\\\frac{81h}{112}&\frac{1161h}{4480}&\frac{17h}{35}&-\frac{729h}{4480}&\frac{27h}{560}&-\frac{29h}{4480}\\\frac{232h}{315}&\frac{64h}{315}&\frac{752h}{945}&\frac{29h}{315}&\frac{8h}{315}&-\frac{4h}{945}\\\frac{725h}{1008}&\frac{2125h}{8064}&\frac{125h}{189}&\frac{3875h}{8064}&\frac{235h}{1008}&-\frac{275h}{24192}\\\frac{27h}{35}&\frac{27h}{280}&\frac{34h}{35}&\frac{27h}{280}&\frac{27h}{35}&\frac{41h}{280}\end{bmatrix},V=\frac{1}{d}\begin{bmatrix}0&0&0&0&0&\frac{19087h}{120960}\\0&0&0&0&0&\frac{1139h}{7560}\\0&0&0&0&0&\frac{137h}{896}\\0&0&0&0&0&\frac{143h}{945}\\0&0&0&0&0&\frac{3715h}{24192}\\0&0&0&0&0&\frac{41h}{280}\end{bmatrix}$$

where; $i\in[1,6], j\in[5,0]$

$$Y_{n+\frac{i}{2}}=\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \end{bmatrix}, Y'_{n+\frac{i}{2}}=\begin{bmatrix} y'_{n+\frac{1}{2}} \\ y'_{n+1} \\ y'_{n+\frac{3}{2}} \\ y'_{n+2} \\ y'_{n+\frac{5}{2}} \\ y'_{n+3} \end{bmatrix}, Y_{n-\frac{j}{2}}=\begin{bmatrix} y_{n-\frac{5}{2}} \\ y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_{n} \end{bmatrix}, Y'_{n-\frac{j}{2}}=\begin{bmatrix} y'_{n-\frac{5}{2}} \\ y'_{n-2} \\ y'_{n-\frac{3}{2}} \\ y'_{n-1} \\ y'_{n-\frac{1}{2}} \\ y'_{n} \end{bmatrix}, \quad F_{n+\frac{i}{2}}=\begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{bmatrix}, \quad F_{n-\frac{j}{2}}=\begin{bmatrix} f_{n-\frac{5}{2}} \\ f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_{n} \end{bmatrix}$$

3. ANALYSIS OF THE SECOND DERIVATIVE HYBRID BLOCK METHOD (SDHBM)

3.1 Order and Error Constant of (SDHBM)

The order and error constant of the proposed method are determined from the following Truncated Talyor Series Expansion of the (SDHBM)

$$\left.\begin{array}{c}
\frac{6031y^{(9)}[x]h^{9}}{464486400}+\frac{18383y^{(10)}[x]h^{10}}{1061683200}+\frac{2013937y^{(11)}[x]h^{11}}{163499212800}+\frac{24225181y^{(12)}[x]h^{12}}{3923981107200}+\frac{8692795649y^{(13)}[x]h^{13}}{3570822807552000} \\
+\frac{5719397923y^{(14)}[x]h^{14}}{7141645615104000}+O[h]^{15} \\
\frac{233y^{(9)}[x]h^{9}}{7257600}+\frac{4961y^{(10)}[x]h^{10}}{116121600}+\frac{173y^{(11)}[x]h^{11}}{5702400}+\frac{116363y^{(12)}[x]h^{12}}{7664025600}+\frac{166820051y^{(13)}[x]h^{13}}{27897053184000} \\
+\frac{438613421y^{(14)}[x]h^{14}}{223176425472000}+O[h]^{15} \\
\frac{9y^{(9)}[x]h^{9}}{179200}+\frac{6129y^{(10)}[x]h^{10}}{91750400}+\frac{95729y^{(11)}[x]h^{11}}{2018508800}+\frac{383247y^{(12)}[x]h^{12}}{16148070400}+\frac{137364519y^{(13)}[x]h^{13}}{14694744064000} \\
+\frac{90301749y^{(14)}[x]h^{14}}{29389488128000}+O[h]^{15} \\
\frac{31y^{(9)}[x]h^{9}}{453600}+\frac{659y^{(10)}[x]h^{10}}{7257600}+\frac{5141y^{(11)}[x]h^{11}}{79833600}+\frac{30841y^{(12)}[x]h^{12}}{958003200}+\frac{22086637y^{(13)}[x]h^{13}}{1743565824000}+\frac{58024619y^{(14)}[x]h^{14}}{13948526592000}+O[h]^{15} \\
\frac{1625y^{(9)}[x]h^{9}}{18579456}+\frac{34625y^{(10)}[x]h^{10}}{297271296}+\frac{77375y^{(11)}[x]h^{11}}{934281216}+\frac{6516325y^{(12)}[x]h^{12}}{156959244288}+\frac{468000125y^{(13)}[x]h^{13}}{28566582460416} \\
+\frac{308271575y^{(14)}[x]h^{14}}{57133164920832}+O[h]^{15} \\
\frac{9y^{(9)}[x]h^{9}}{89600}+\frac{27y^{(10)}[x]h^{10}}{204800}+\frac{727y^{(11)}[x]h^{11}}{7884800}+\frac{1431y^{(12)}[x]h^{12}}{31539200}+\frac{2013681y^{(13)}[x]h^{13}}{114802688000}+\frac{5187363y^{(14)}[x]h^{14}}{918421504000}+O[h]^{15} \\
\frac{275y^{(9)}[x]h^{8}}{6193152}+\frac{55211y^{(10)}[x]h^{9}}{928972800}+\frac{39379y^{(11)}[x]h^{10}}{928972800}+\frac{41431y^{(12)}[x]h^{11}}{1946419200}+\frac{16490711y^{(13)}[x]h^{12}}{1961990553600}+\frac{1647295913y^{(14)}[x]h^{13}}{595137134592000} \\
+\frac{1688619833y^{(15)}[x]h^{14}}{2142493684531200}+O[h]^{15} \\
\frac{y^{(9)}[x]h^{8}}{30240}+\frac{361y^{(10)}[x]h^{9}}{8294400}+\frac{3557y^{(11)}[x]h^{10}}{116121600}+\frac{12961y^{(12)}[x]h^{11}}{851558400}+\frac{571y^{(13)}[x]h^{12}}{95800320}+\frac{8059109y^{(14)}[x]h^{13}}{4132896768000} \\
+\frac{14793319y^{(15)}[x]h^{14}}{26781171056640}+O[h]^{15} \\
\frac{9y^{(9)}[x]h^{8}}{229376}+\frac{603y^{(10)}[x]h^{9}}{11468800}+\frac{431y^{(11)}[x]h^{10}}{11468800}+\frac{9547y^{(12)}[x]h^{11}}{504627200}+\frac{60477y^{(13)}[x]h^{12}}{8074035200}+\frac{18169609y^{(14)}[x]h^{13}}{7347372032000} \\
+\frac{6222883y^{(15)}[x]h^{14}}{8816846438400}+O[h]^{15} \\
\frac{y^{(9)}[x]h^{8}}{30240}+\frac{313y^{(10)}[x]h^{9}}{7257600}+\frac{109y^{(11)}[x]h^{10}}{3628800}+\frac{1571y^{(12)}[x]h^{11}}{106444800}+\frac{4379y^{(13)}[x]h^{12}}{766402560}+\frac{8593579y^{(14)}[x]h^{13}}{4649508864000} \\
+\frac{4334483y^{(15)}[x]h^{14}}{8369115955200}+O[h]^{15} \\
\frac{275y^{(9)}[x]h^{8}}{6193152}+\frac{325y^{(10)}[x]h^{9}}{5308416}+\frac{1675y^{(11)}[x]h^{10}}{37158912}+\frac{12745y^{(12)}[x]h^{11}}{544997376}+\frac{748655y^{(13)}[x]h^{12}}{78479622144}+\frac{190605y^{(14)}[x]h^{13}}{58778976256} \\
+\frac{81573665y^{(15)}[x]h^{14}}{85699747381248}+O[h]^{15} \\
-\frac{9y^{(10)}[x]h^{9}}{716800}-\frac{27y^{(11)}[x]h^{10}}{1433600}-\frac{67y^{(12)}[x]h^{11}}{4505600}-\frac{129y^{(13)}[x]h^{12}}{15769600}-\frac{1610347y^{(14)}[x]h^{13}}{459210752000}-\frac{228453y^{(15)}[x]h^{14}}{183684300800}+O[h]^{15}
\end{array}\right\} \quad (3.1)$$

Equation (3.1) shows that the corresponding scheme is of order 7, 7, 7, 7, 7, 7, 6, 6, 6, 6, 6, 7 with error constants respectively, as follows:

$$\left[\frac{6031}{464486400}, \frac{233}{7257600}, \frac{9}{179200}, \frac{31}{453600}, \frac{1625}{18579456}, \frac{9}{89600}, \frac{275}{6193152}, \frac{1}{30240}, \frac{9}{229376}, \frac{1}{30240}, \frac{275}{6193152}, -\frac{9}{716800}\right]^T.$$

Consistency: The block method (2.19) is consistent because its order is more than 1.

Zero Stability of (SDHBM):-

The first characteristic polynomial of (2.19) is:

$\rho(r)$ $=det[rI - A]$

$$\rho(r) = det\left[\begin{bmatrix} r & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 & 0 & r \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right],$$

$$\begin{vmatrix} r & 0 & 0 & 0 & 0 & -1 \\ 0 & r & 0 & 0 & 0 & -1 \\ 0 & 0 & r & 0 & 0 & -1 \\ 0 & 0 & 0 & r & 0 & -1 \\ 0 & 0 & 0 & 0 & r & -1 \\ 0 & 0 & 0 & 0 & 0 & r-1 \end{vmatrix} = (r-1)r^5$$

Then,$\rho(r) = (r-1)r^5 = 0$ means that $r = 0 \; or \; r = 1$.

So, the method is zero stable.

To investigate the stability region, we considered the Boundary Locus Technique together with the test equation given as:

$y'' = \lambda^2 y$

where $\lambda \in \mathbb{C}$

Now from the block method:

$$\left.\begin{aligned} IY_{n+\frac{i}{2}} &= AY_{n-\frac{j}{2}} + hBY'_{n-\frac{j}{2}} + h^2\left(\frac{1}{s}CF_{n+\frac{i}{2}} + \frac{1}{d}NF_{n-\frac{j}{2}}\right) \\ GY'_{n+\frac{i}{2}} &= TY'_{n-\frac{j}{2}} + h\left(\frac{1}{s}EF_{n+\frac{i}{2}} + \frac{1}{d}VF_{n-\frac{j}{2}}\right) \end{aligned}\right\}$$

Test equation is used to transform the above block method to:

$$\left.\begin{aligned} IY_{n+\frac{i}{2}} &= AY_{n-\frac{j}{2}} + hBY'_{n-\frac{j}{2}} + \frac{1}{s}Ch^2\lambda^2 Y_{n+\frac{i}{2}} + \frac{1}{d}Nh^2\lambda^2 Y_{n-\frac{j}{2}} \\ GY'_{n+\frac{i}{2}} &= TY'_{n-\frac{j}{2}} + \frac{1}{s}Eh\lambda^2 Y_{n+\frac{i}{2}} + \frac{1}{d}V\lambda^2 hY_{n-\frac{j}{2}} \end{aligned}\right\}$$

$$\left.\begin{aligned} IY_{n+\frac{i}{2}} &= AY_{n-\frac{j}{2}} + hBY'_{n-\frac{j}{2}} + \frac{1}{s}CzY_{n+\frac{i}{2}} + \frac{1}{d}NzY_{n-\frac{j}{2}} \\ GY'_{n+\frac{i}{2}} &= TY'_{n-\frac{j}{2}} + \frac{1}{s.h}EzY_{n+\frac{i}{2}} + \frac{1}{d.h}VzY_{n-\frac{j}{2}} \end{aligned}\right\} \quad (3.2)$$

We form a coupled block method from (3.2) as:

$$\begin{bmatrix} I - \frac{z}{s}C & 0 \\ -\frac{z}{s.h}E & I \end{bmatrix} \begin{bmatrix} Y_{n+\frac{i}{2}} \\ Y'_{n+\frac{i}{2}} \end{bmatrix} = \begin{bmatrix} A + zN & hB \\ \frac{z}{h.d}V & A \end{bmatrix} \begin{bmatrix} Y_{n-\frac{j}{2}} \\ Y'_{n-\frac{j}{2}} \end{bmatrix} \quad (3.3)$$

$$M(z) = \begin{bmatrix} I - \frac{z}{s}C & 0 \\ -\frac{z}{s.h}E & I \end{bmatrix}^{-1} \begin{bmatrix} A + zN & hB \\ \frac{z}{h.d}V & A \end{bmatrix} \quad (3.4)$$

$Note: Matrix \; A = T \; and \; I = G$

$M(z)$ is the amplification matrix which is used to obtain the stability plots as follows:

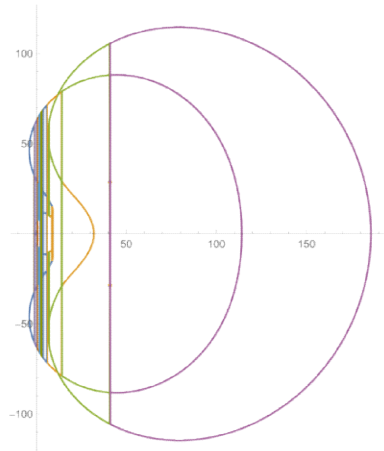


Figure1: Stability Region of SDHBM for$d = 0.9$; $s = -1.2$
Figure1 shows that SDHBM is A(α) stable.

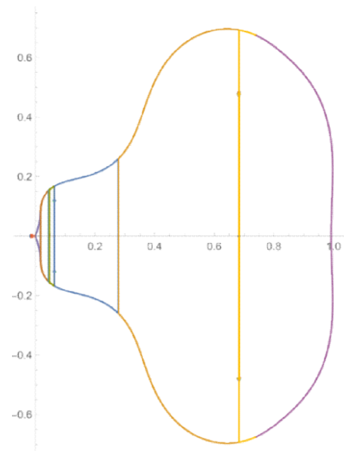


Figure.2: Stability Region of SDHBM for$d = 1; s = 1$
Figure.2 shows that SDHBM is A-stable.

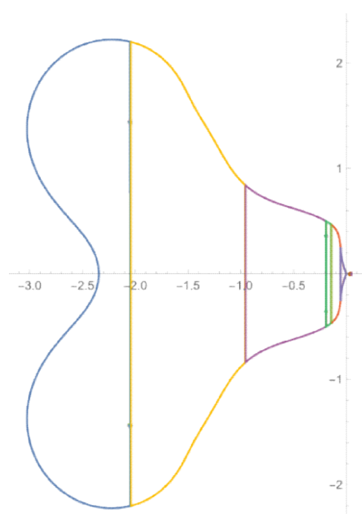


Figure.3: Stability Region of SDHBM for$d = -0.78; s = -0.06$
Figure 3 has shown that SDHBM is stable in a defined region$[-2.5, 0]$

Numerical Experiment:-
For experiment, we considered the following problems.
Problem 4. 1:$y'' = y';\ y(0) = 0, y'(0) = -1,\ h\ =\ 0.1 x\epsilon[0, 2]$
Exact solution: $y(x) = 1 -\ e^x$(Kayode & Adegboro 2018).

Problem 4.2:$y'' = x(y')^2, y(0)\ = 1,\ y'(0) = 0.5, h = 0.01$, $x\epsilon[0, 2]$
Exact solution: $y = 1 + \frac{1}{2} ln \left(\frac{2+x}{2-x}\right)$Adeyeye et al (2017)
Problem 4.3: $y'' - 2y^3 = 0, y(1) = 1, y'(1) = -1, h = \frac{1}{10},\qquad x\epsilon[0, 2]$
Exact solution:$y(x) = \frac{1}{x}$ Kamarun et al (2023)

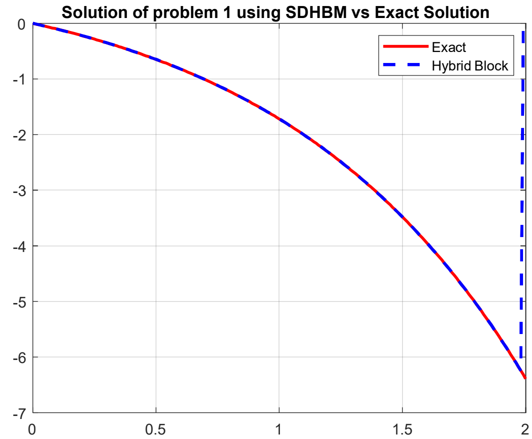


Figure 4. Solution to problem 4.1
Figure 4. is the solution to problem 4.1 which is solved directly with SDHBM and the result is compared with the exact solution and it shows competitiveness.

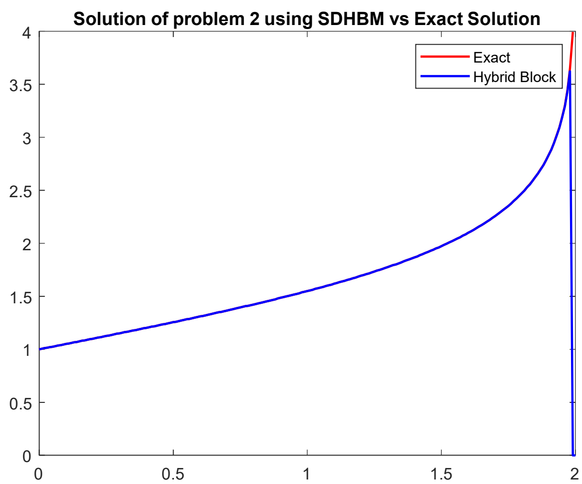


Figure 5 is a solution to problem 4.2 which is solved with SDHBM directly when the method is A-stable and the result closely approximate the exact solution. That is, zero error or negligibly small error.

Table 1: Comparison of errors between the proposed scheme and Kamarun et al (2023)

x	Error from the proposed Method	Error from Kamarun et al (2023)
1.10	5.4903e-12	2.810252e-11
1.20	1.9997e-11	4.390854e-11
1.30	4.0977e-11	2.088394e-08

1.40	6.6200e-11	4.254005e-08
1.50	9.5770e-11	6.852856e-08
1.60	1.2963e-10	9.633702e-08
1.70	1.6674e-10	1.269875e-07
1.80	2.0878e-10	1.602706e-07
1.90	2.5426e-10	1.966604e-07
2.00	3.0476e-10	2.363168e-07

Table 1 Shows the error composition of the proposed method and Kamarun et al (2023).

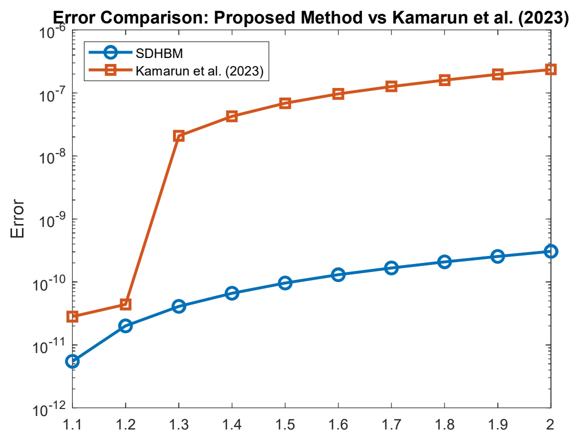


Figure 6is the error plot of the proposed method and Kamarun et al (2023)

Discussion of Results:-

The graphical results presented in Figures 4 and 5 for Test Problems 4 and 5, respectively, show that the numerical solutions closely overlap with the exact solutions throughout the integration interval. This indicates the high accuracy and consistency of the proposed method in approximating the exact solutions. Furthermore, Table.1 presents the error comparison between the proposed method and the method developed by Kamarun et al.(2023) for Test Problem 4.3. The corresponding error plot, shown in Figure 6 demonstrates that the proposed method produces smaller errors, which therefore shows its efficiency, and reliability for solving IVPs in ODEs.The close agreement between both solutions further validates the accuracy and stability of the proposed method for solving the considered ordinary differential equations.

Conclusion:-

We have introduced the Second Derivative Hybrid Block Method (SDHBM) for the direct solution of second-order initial value problems in ordinary differential equations. The method was tested on some IVPs, and the results obtained were compared with the exact solutions as well as with the error results reported by Kamarun et al. (2023). The comparisons show that the proposed method is competitive. These results suggest that the method provides a reliable alternative solver for second-order IVPs in ODEs, with potential applications to more complex systems arising from real-world problems.

References:-

1. Abdulazeez, K. J. (2024).Two-Step Hybrid Block Method for Solving SecondOrder Initial Value Problem of Ordinary DifferentialEquations.Earthline Journal of Mathematical Sciences Volume 14(5):1047-1065
2. Adeyefa, E.O. & Olagunju S.A. (2021). Hybrid Block Method for Direct Integration of First , Second and Third Order IVPs. Journal of Science and Engineering 18(1), 1 -8
3. Adeyeye, O., and Omar, Z. (2017). 4-step 5-point hybrid block method for the direct solution of second order initial value problems, Journal of Mathematics and Computer Science17, 527 – 534.
4. Areo, E. A., & Adeniyi, R. B. (2014). Block implicit one-step method for the numerical integration of initial value problems in ordinary differential equations. International Journal of Mathematics and Statistics Studies, 2(3), 4-13.
5. Awoyemi, D. O (1999). A Class of Continuous Methods for the Solution of General Second Order Ordinary Differential Equations. International Journal of Computational Mathematics, Vol 72 pp 29-37.
6. Bolaji, B. A. (2017). A zero-stable hybrid linear multistep method for the numerical solution of first order ordinary differential equations. American Journal of Engineering and Natural Sciences (AJENS), 1(2), 1-7.
7. Dahlquist, G. (1956). Convergence and stability in the numerical integration of ordinary differential equations. MATHEMATICA SCANDINAVICA, 4, 33–53.

8. Jain, M. K., Iyengar, S. R. K., & Jain, R. K. (2012). Numerical Methods for Scientific and Engineering Computation (6th ed.). New Age International Publishers.
9. Kamarun, H.M., Adeyeye, O., & Omar, Z. (2023). Two-Step Hybrid Block Method with Generalized Two Off-Step Points Within Each Step for solving second order initial value problems. Journal of International Information and Engineering Technology Association 10(2), 433- 441.
10. Kayode S.J. Adegboro, J. O. (2018).Predictor-Corrector Linear Multistep Method forDirect Solution of Initial Value Problems of SecondOrder Ordinary Differential Equations Asian Journal of Physical and Chemical Sciences 6(1): 1-9
11. Okuonghae, R. and Ikhile, M. (2014). A class of hybrid linear multistep methods with A(α)-stable properties for stiff IVPs in ODEs. Journal of Numerical Mathematics 8(4), 441-469.
12. Sagir, A. M. (2014). Numerical treatment of block method for the solution of ordinary differential equations. International Journal of Bioengineering and Life Sciences, 8(2), 259-263.
13. Tadema, A. J. (2024). Numerical solution of second order ordinary differential equations. Computational Algorithms and Numerical Dimensions, 3(4), 277-290.
14. Taiwo, E.O., Moshood, L.O. & Ogunniran M.O. (2023). Effectiveness of a One-fifth Hybrid Block Approach for Second Order Ordinary Differential Equations. Istanbul Journal of Mathematics, 1(2), 86 – 95.
15. Workineh, Y., Mekonnen, H. and Belew, B. (2024). Numerical methods for solving second-order initial value problems of ordinary differential equations with Euler and Runge-Kutta fourth-order methods. Frontiers of Applied Mathematics and Statistics.